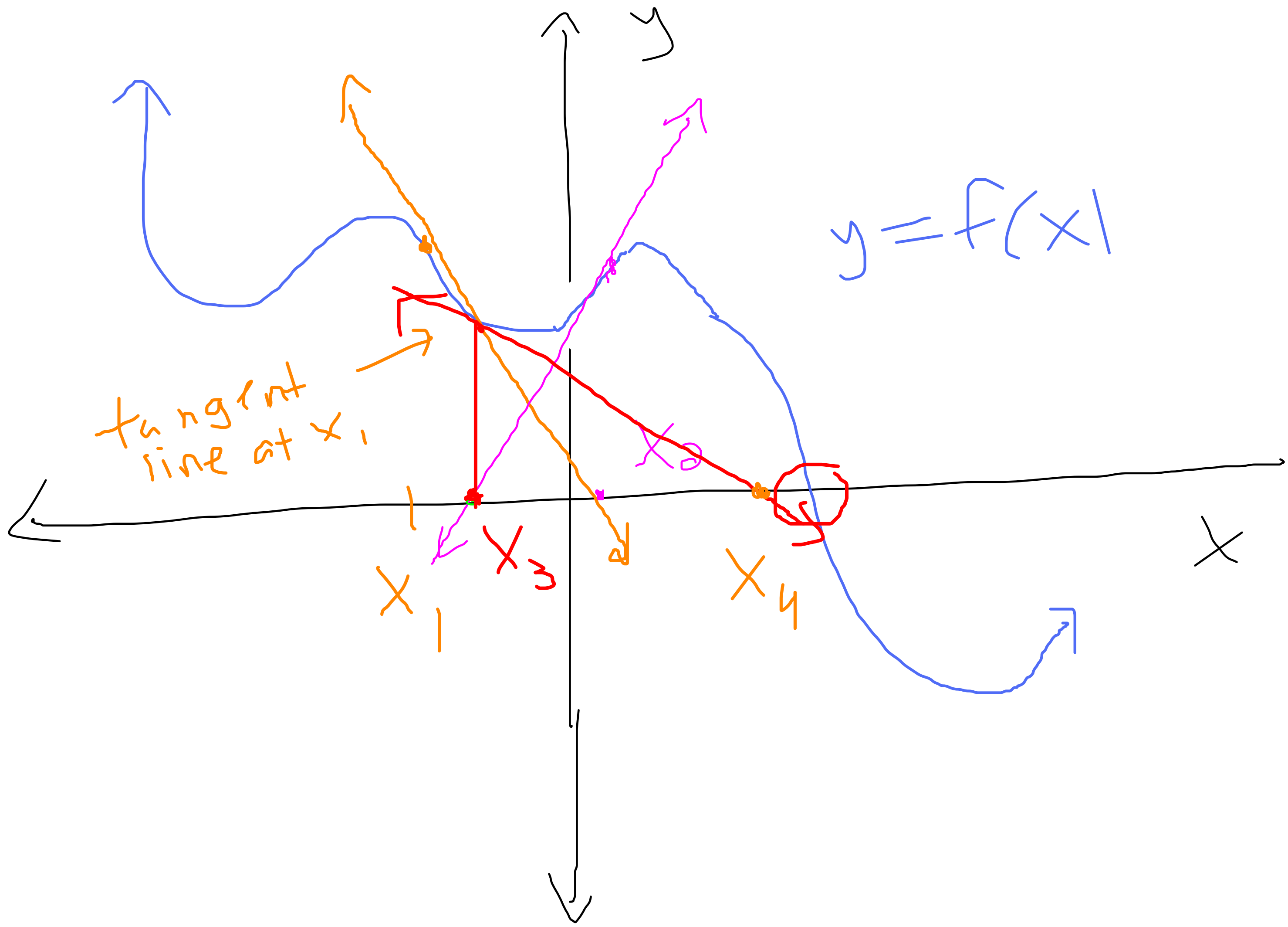
Announcement

1) HW 4 due Sunday evening

Picture : (Newtons method)



$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

Your hope is that

$$\lim_{n \to \infty} x_n = \text{a zero of } f.$$

There are problems!

Problems: 1) What if $f'(x_i) = 0$ for some i?

2) What if x_{i+1} is a worse approximation than x_i?

Solution for 1): Choose a different value for x_1!

Might as well try the same for 2).

Example 1: $f(x) = x^3 - x + 1$

$f(4) = 64 - 4 + 1 = 61 > 0$

$f(-2) = -8 + 2 + 1 = -5 < 0$.

By the Intermediate Value Theorem, f has a zero in $(-2, 4)$.

Approximate the zero using Newton's Method. Choose x_1 in $(-2, 4)$

$$x_1 = \frac{1}{\sqrt{3}}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$f'(x) = 3x^2 - 1$$

$$f'(x_1) = f'\left(\frac{1}{\sqrt{3}}\right) = 3 \cdot \frac{1}{3} - 1 = 0$$

There is no x_2, so the method fails. Try a different choice for x_1,

$x_1 = 0$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$f(x) = x^3 - x + 1$$

$$f'(x) = 3x^2 - 1$$

$$x_2 = 0 - \frac{1}{-1} = 1$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

Keep going . . .

Example 2:

$$f(x) = x^3 - 2x + 2$$

use $x_1 = 0$

in Newton's method.

$$f'(x) = 3x^2 - 2$$

$$x_2 = 0 - \frac{f(0)}{f'(0)} = -\frac{2}{-2} = 1$$

$$x_3 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{1} = 0$$

$$= x_1$$

This is an example of the second kind of failure, where the sequence x_n does not "tend to" anything — pick a different x_1!

Example 3: $\sqrt[3]{17}$ to 4 decimal places.

$$2^3 = 8 < \left(\sqrt[3]{17}\right)^3 = 17$$
$$< 3^3 = 27$$

$\sqrt[3]{17}$ is between 2 and 3.

Use Newton's Method

Use on $f(x) = x^3 - 17$.

Pick a point in $(2,3)$

$x_1 = 2.5$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$f'(x) = 3x^2$$

$$x_2 = \frac{5}{2} - \frac{\frac{125}{8} - 17}{3 \cdot \frac{25}{4}} \cdot \frac{8}{8} (=1)$$

$$= \frac{5}{2} - \frac{125 - 17 \cdot 8}{6 \cdot 25} = \frac{5}{2} - \frac{-11}{150}$$

$$= \frac{375}{150} + \frac{11}{150}$$

$$= \frac{386}{150} \approx 2.573\overline{3}$$

Need one more decimal place

$$x_3 = \frac{386}{150} - \frac{f\left(\frac{386}{150}\right)}{f'\left(\frac{386}{150}\right)}$$

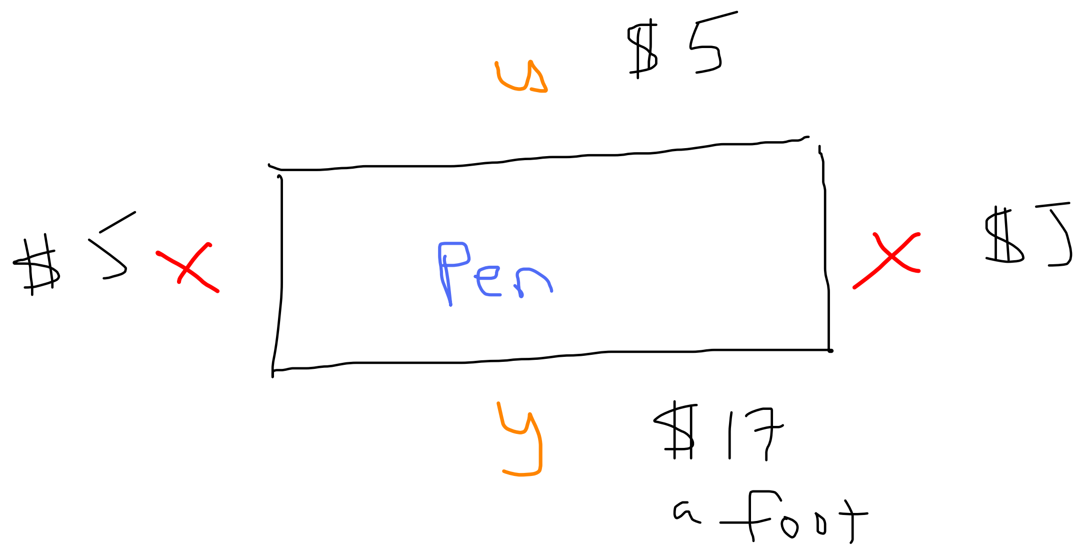
will be good enough!

Another Optimization Problem

Farmer Bocephus is building a rectangular pen. The pen needs a more solid side facing his house than the other sides.

If the cheaper sides cost $5 a foot and the more expensive side $17 a foot, how much of each material should he buy to minimize costs if the pen must have a fixed area of 300 square feet?

Picture:



We know $x \cdot y = 300$.

Want to minimize cost

Cost of one side of length x is \$5. There are 2 such sides, so this gives $\$2 \cdot 5 \cdot x$

The remaining sides of length y vary in cost; one costs $\$17y$ and the other $\$5y$.

Total cost = sum of all costs

$$= C(x,y) = 2 \cdot 5 \cdot x + 5 \cdot y + 17 \cdot y$$

$$= 10x + 22y$$

Use the fact that $300 = xy$ to get a one-variable cost.

Then $x = \frac{300}{y}$, so substituting into $C(x,y)$, we get

$$C(y) = 10 \cdot \left(\frac{300}{y}\right) + 22y$$

$$= \frac{3000}{y} + 22y$$

Differentiate, set equal to zero.

$$C(y) = 3000\, y^{-1} + 22y$$

$$C'(y) = -3000\, y^{-2} + 22 = 0$$

We get

$$3000y^{-2} = 22$$

so $y^{-2} = \frac{22}{3000}$

$$\frac{1}{y^2} = \frac{22}{3000}, \text{ and}$$

$$y^2 = \frac{3000}{22}$$

$$y = \sqrt{\frac{3000}{22}}$$

(negative not physically possible)

Check this is a minimum using 2^{nd} derivative test.

$$C'(y) = -3000\,y^{-2} + 22$$

$$C''(y) = 6000\,y^{-3}$$

$$C''\left(\sqrt{\frac{3000}{22}}\right) = \frac{6000}{\left(\sqrt{\frac{3000}{22}}\right)^3} > 0$$

We do have a minimum!

Dimensions are

$$\boxed{\begin{aligned} y &= \sqrt{\frac{3000}{22}}\ \text{ft} \\ x &= \frac{300}{y} = \frac{300\sqrt{22}}{\sqrt{3000}}\ \text{ft.} \end{aligned}}$$

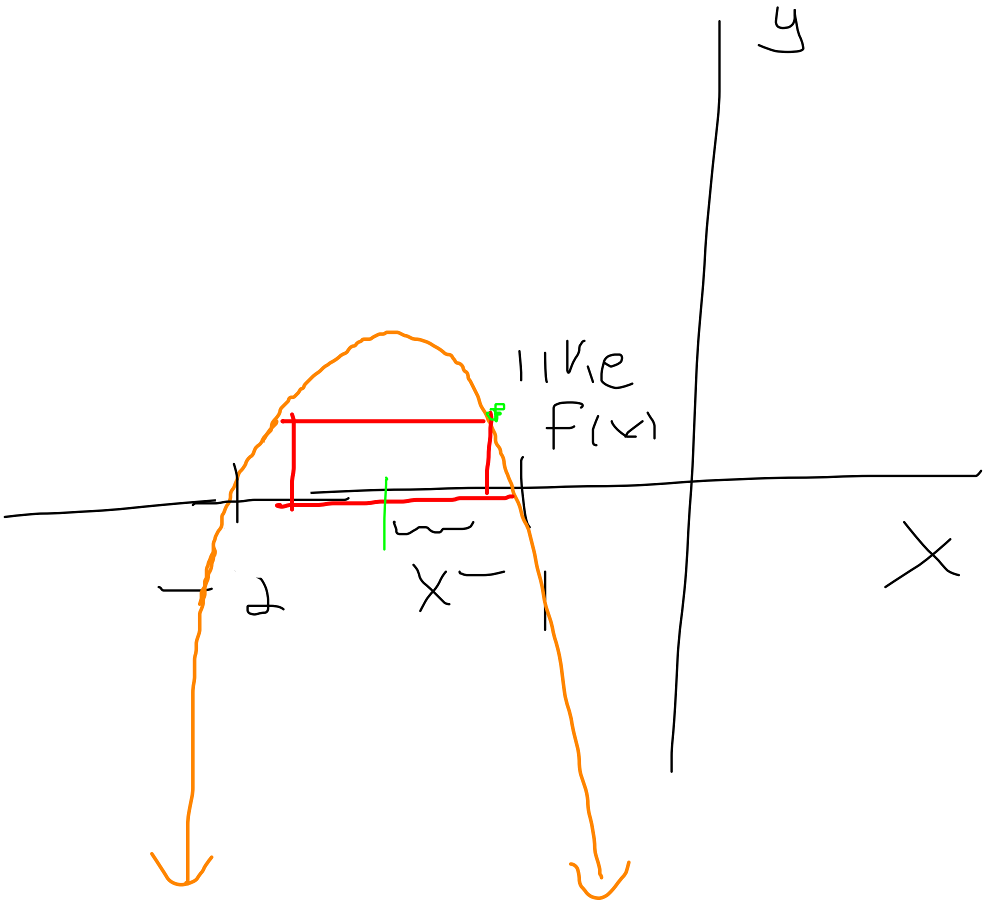
Another Optimization

Sasquatch is walking a rectangular path with vertices on the x-axis and above the x-axis on the parabola

$$f(x) = -x^2 - 3x - 2.$$

Find the rectangle with maximum perimeter.

Picture



length = $2x$

height : $f(x)$

perimeter $= 4x + 2f(x)$

maximize.